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Primitive Ideals and Orbital Integrals in Complex Exceptional Groups

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1. INTRODUCTION

In [2], two related problems were studied for complex classical groups: determination of the primitive spectrum of the enveloping algebra (as a set), and Fourier inversion of unipotent orbital integrals. The first of these was solved completely (extending work of Joseph and others in $SL(n, \mathbb{C})$ and small groups). Although our techniques presumably solve the second as well, up to determination of normalizing constants, we carried out the calculations only for special unipotent classes (see [18]). We had, of course, tried to apply the same methods to the exceptional groups, but they seem to be inadequate. Since that work was done, however, Brylinski and Kashiwara and Beilinson and Bernstein have established the Kazhdan–Lusztig conjecture, giving character formulas for irreducible highest weight modules [3, 7]. Because of a conjecture of Joseph proved in [24], this determines in principle the primitive ideals with a fixed regular integral infinitesimal character: they are in one-to-one correspondence with what Kazhdan and Lusztig call left cells in the Weyl group. However, the algorithm given by Kazhdan and Lusztig to compute these cells is enormously complicated, requiring one to compute roughly $|W|^2$ polynomials of degrees on the order of half the number of positive roots. This is very unsatisfactory. However, it

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is a consequence of the structure of the algorithm that there is an inclusion-reversing involution of the set of primitive ideals in question (Corollary 2.24); this fact, which has been noticed by many people, confirms a conjecture of Borho and Jantzen. It is amazingly powerful, and leads easily to a rough determination of the primitive ideals in the exceptional enveloping algebras. Our results are less satisfactory than in the classical case, but at least Theorem D of [2] can now be stated in general. Without recalling all the notation (see Section 2), this is

THEOREM 1.1 (Theorem 2.29 below). *Suppose $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, $\lambda \in \mathfrak{h}^*$ is regular, W_λ is the integral Weyl group, and $I \in \mathrm{Prim}_\lambda U(\mathfrak{g})$. Then Joseph's Goldie rank representation $\sigma(I) \in \hat{W}_\lambda$ [16] is a special representation of W_λ in the sense of Lusztig [18]; and all special representation occur. In particular,*

$$|\mathrm{Prim}_\lambda U(\mathfrak{g})| = \sum_{\substack{\sigma \in \hat{W}_\lambda \\ \sigma \text{ special}}} \dim \sigma.$$

Just as in [2], there are consequences for the Fourier inversion of unipotent orbital integrals; this is discussed in Section 4. Briefly, we treat all the unipotent orbits in exceptional groups except for three in E_8; and for each of these, the differential operator which is needed is shown to be some linear combination of two specified operators.

It is a pleasure to thank G. Lusztig yet again for his continuing assistance in the formulation and proofs of these results. Our calculations are based on a complete knowledge of how representations of exceptional Weyl groups restrict to parabolic subgroups; for this we have used tables prepared by Alvis (see [26]; this unpublished manuscript is available from Alvis).

2. General Results and Notation

For this section, $\mathfrak{g}$ can be any reductive complex Lie algebra. Some basic notation will be carrried over from [2], although we will try to repeat most definitions. Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, and a regular weight $\lambda \in \mathfrak{h}^*$. Write

$$\chi_\lambda : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C} \tag{2.1}$$

for the corresponding character of the center $\mathfrak{Z}(\mathfrak{g})$ of the enveloping algebra of $\mathfrak{g}$; χ_λ depends only on the $W(\mathfrak{g}, \mathfrak{h})$ orbit of λ. Define

$\operatorname{Prim}_\lambda U(\mathfrak{g}) = \{I \subseteq U(\mathfrak{g})$ (two-sided) primitive ideal, $I \cap \mathfrak{Z}(\mathfrak{g}) = \ker \chi_\lambda\}$

$$\begin{aligned}
&R_\lambda = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \langle \check{\alpha}, \lambda \rangle \in \mathbb{Z}\} \qquad \left(\text{here } \check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle}\right),\\
&R_\lambda^+ = \{\alpha \in R_\lambda \mid \langle \alpha, \lambda \rangle > 0\},\\
&\Pi_\lambda = \text{simple roots of } R_\lambda^+,\ S_\lambda = \{\text{reflections } s_\alpha \mid \alpha \in \Pi_\lambda\},\\
&W_\lambda = W(R_\lambda), \text{ the \textit{integral Weyl group}.}
\end{aligned} \tag{2.2}$$

The Bruhat order on W_λ is written $\leqslant$.

Now choose a positive root system

$$\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h}) \supseteq -R_\lambda^+; \tag{2.3}$$

that is, we assume λ is *negative*. As was first shown in [13], the particular choice of Δ^+ will not affect any of the results on primitive ideals; this is also clear from [14]. Put

$$\begin{aligned}
&\text{(a)}\quad \mathfrak{b} = \mathfrak{h} + \mathfrak{n}, \qquad \Delta(\mathfrak{n}, \mathfrak{h}) = \Delta^+,\\
&\text{(b)}\quad M(\mu) = U(\mathfrak{g}) \otimes \mathbb{C}_{\mu-\rho},\\
&\text{(c)}\quad L(\mu) = \text{irreducible quotient of } M(\mu),\\
&\text{(d)}\quad I(\mu) = \text{annihilator of } L(\mu) \text{ in } U(\mathfrak{g}),\\
&\text{(e)}\quad I(w) = \text{annihilator of } L(w\lambda) \qquad (w \in W_\lambda),\\
&\text{(f)}\quad L(w\lambda) = \textstyle\sum_{y \in W_\lambda} a_{y,w} M(y\lambda) \mid (w \in W_\lambda).
\end{aligned} \tag{2.4}$$

Here the second equation in (a) defines $\mathfrak{n}$, so that $\mathfrak{b}$ is a Borel subalgebra; $M(\mu)$ is a Verma module; $I(\mu)$ is a primitive ideal; and the integers $a_{y,w}$ of (f) are defined by this equation (in some Grothendieck group), the formal character of $L(w\lambda)$.

THEOREM 2.5 [9]. *The map* $w \to I(w) = \operatorname{Ann} L(w\lambda)$ (cf. (2.4(e))) *from* W_λ *to* $\operatorname{Prim}_\lambda U(\mathfrak{g})$, *is surjective.*

The best answer to the problem of determining $\operatorname{Prim}_\lambda U(\mathfrak{g})$ as a set is a simple description of the fibers of this map. When W_λ is classical, this was done in a fairly reasonable way in [2]. As indicated in the introduction, we are less successful with the exceptional groups; but still one can say a lot.

THEOREM 2.6 [16]. *Fix* $\lambda \in \mathfrak{h}^*$ *regular, and notation* (2.1)–(2.4); *and choose a dominant regular element* $x \in \mathfrak{h}$. *For* $w \in W_\lambda$, *set*

$$a(w) = |\Delta^+| - GK\ dimension\ (L(w\lambda)),$$

$$r(w) = \sum_{y \in W_\lambda} a_{y,w}(y^{-1}x)^{a(w)},$$

a polynomial function on $\mathfrak{h}^*$; *here* $a_{y,w}$ *is defined by* (2.4)(f).

(a) $I(w) = I(w')$ *if and only if* $r(w) = c \cdot r(w')$, *for some positive constant* c.

(b) *If* $r'(w)$ *is defined similarly using* $x' \in \mathfrak{h}$, *then* $r'(w) = c'r(w)$ *for some positive constant* c'. *If* $r_m(w)$ *is defined similarly, with* m *replacing* $a(w)$, *then* $r_m(w) = 0$ *for* $m < a(w)$.

(c) $r(w)$ *is a* W_λ*-harmonic polynomial, and generates an irreducible representation* $\sigma(w) \in \hat{W}_\lambda$ (*with respect to the action of* W_λ *on* $S(\mathfrak{h})$).

(d) $a(w)$ *is the lowest degree in which* $\sigma(w)$ *occurs in* $S(\mathfrak{h})$, *and it occurs exactly once in this degree.*

(e) *As* w *runs over* W_λ, *the* $r(w)$ *gives bases of the various representations* $\sigma(w)$, *subject only to the repetitions described by* (a) *above.*

DEFINITION 2.7. The polynomial $r(w) \in S(\mathfrak{h})$ described in Theorem 2.6 is called the *character polynomial* of $L(w\lambda)$. If $I = I(w) \in \mathrm{Prim}_\lambda U(\mathfrak{g})$, put

$$\sigma(I) = \sigma(w) \in \hat{W}_\lambda$$

[cf. Theorem 2.6(c)], the *Goldie rank representation* for I.

DEFINITION 2.8 [15]. Identify the group algebra $\mathbb{C}[W_\lambda]$ with the group of formal complex combinations

$$\sum_{w \in W_\lambda} c_w M(w\lambda).$$

Make $W_\lambda \times W_\lambda$ act by the regular representation; thus

$$(w_1, w_2) \cdot \sum c_w M(w\lambda) = \sum c_w M(w_1 w w_2^{-1} \lambda).$$

We regard the modules $L(w\lambda)$ as elements of $\mathbb{C}[W_\lambda]$. A subspace V of $\mathbb{C}[W_\lambda]$ is called *a-basal* if it is the span of the $L(w\lambda)$ contained in it.

PROPOSITION 2.9 [15, 24]. *In the setting of Theorem* 2.6, *suppose* $w_1, w_2 \in W_\lambda$. *Then the following conditions are equivalent*:

(a) $I(w_1) \subseteq I(w_2)$

(b) $L(w_2\lambda)$ *belongs to the minimal a-basal subspace of* $\mathbb{C}[W_\lambda]$ *containing* $L(w_1\lambda)$ *and invariant by the left action of* W_λ.

(c) *There are sequences* $\{z_0,..., z_n\} \subseteq W_\lambda$, $\{s_1,..., s_n\} \subseteq S_\lambda$, *such that* $z_0 = w_1$, $z_n = w_2$; *and for each* $i = 1,..., n$,

$$(s_i, 1) \cdot L(z_{i-1}\lambda) = \sum c_z^i L(z\lambda),$$

with $c_{z_i}^i \neq 0$.

(d) *There is a finite dimensional representation* F *of* $\mathfrak{g}$, *such that* $L(w_2^{-1}\lambda)$ *is subquotient of* $L(w_1^{-1}) \otimes F$.

Proof. Since S_λ generates W_λ, statement (c) is a reformulation of (b). The equivalence of (a), (b), and (d) is buried at various depths in [15] and [24]; the main thing needed in addition to [24] is the relation between characters of highest weight modules and Harish-Chandra modules, proved by Joseph in [14] (and also by Enright and by I. Bernstein and S. Gelfand). The formulations of the next few results are largely due to Lusztig. Q.E.D.

DEFINITION 2.10. Suppose $w_1, w_2 \in W_\lambda$. Under any of the conditions in Proposition 2.9, we say $w_1 \leqslant_L w_2$. We say $w_1 \leqslant_R w_2$ if and only if $w_1^{-1} \leqslant_L w_2^{-1}$. Put

$$\bar{\mathscr{C}}_w^L = \{w' \in W_\lambda \mid w' \geqslant_L w\} = \{w' \in W_\lambda \mid I(w') \supseteq I(w)\}.$$

Similarly, we define $\bar{\mathscr{C}}_w^R$. The smallest preorder relation containing $\leqslant_L$ and $\leqslant_R$ is called $\leqslant_{LR}$; that is, $w_1 \leqslant_{LR} w_2$ if and only if there is a sequence $\{z_0,..., z_n\} \subseteq W_\lambda$, with $w_1 = z_0$, $w_2 = z_n$, and either $z_{i-1} \leqslant_L z_i$, or $z_{i-1} \leqslant_R z_i$, for all i between 1 and n. Define $\bar{\mathscr{C}}^{LR}$ as above. The sets $\bar{\mathscr{C}}_w^L$, etc., are called the *left, right, etc. cones over* w. Define $w_1 \approx_L w_2$ if and only if $I(w_1) = I(w_2)$; equivalently (by Proposition 2.9), if $w_1 \leqslant_L w_2 \leqslant_L w_1$. Similarly, define $\approx_R$, $\approx_{LR}$. The equivalence classes of $\approx_L$ (respectively, $\approx_R$, $\approx_{LR}$) are called *left* (respectively *right, two-sided*) *cells*. The cells containing w are written $\mathscr{C}_w^L$, $\mathscr{C}_w^R$, $\mathscr{C}_w^{LR}$. Set

$$\bar{V}_w^L = \bigoplus_{w' \in \bar{\mathscr{C}}_w^L} \mathbb{C} \cdot L(w\lambda) \subseteq \mathbb{C}[W_\lambda];$$

(notation (2.8))

$$K_w^L = \bigoplus_{\substack{w' \in \bar{\mathscr{C}}_w^L \\ w' \notin \mathscr{C}_w^L}} V_{w'}^L \subseteq \bar{V}_w^L,$$

$$V_w^L = \bar{V}_w^L / K_w^L.$$

Similarly, we define all of these objects with a superscript R or LR.

COROLLARY 2.11 [15]. *The subspaces $\bar{V}^L_w$ and K^L_w are invariant under the left action of W_λ on $\mathbb{C}[W_\lambda]$; so V^L_w carries a natural representation of W_λ. Similarly, V^R_w carries a representation of W_λ, and V^{LR}_w a double representation. We have*

$$\bigoplus_{\text{cells } \mathscr{C}_w} V_w \cong \mathbb{C}[W_\lambda],$$

with any choice of superscript L, R or LR; this is as a left, right, or double representation accordingly.

The last assertion is the only one which is not clear from Proposition 2.9 and the definitions, and it follows from the fact that the associated graded representation of a filtered representation of a finite group is isomorphic to it (though not naturally).

DEFINITION 2.12. Suppose $I = I(w) \in \operatorname{Prim}_\lambda U(\mathfrak{g})$. The *left cell representation* of W_λ associated to I or w is V^L_w; the *double cell representation* is V^{LR}_w. Suppose $\sigma_1, \sigma_2 \in \hat{W}_\lambda$. We say $\sigma_1 \leqslant_{LR} \sigma_2$ if $\sigma_1 \otimes \sigma_1$ (the double representation) occurs in V^{LR}_w, and $\sigma_2 \otimes \sigma_2$ in $\bar{V}^{LR}_w$. Thus $\sigma_1 \approx_{LR} \sigma_2$ if and only if $\sigma_1 \otimes \sigma_1$ and $\sigma_2 \otimes \sigma_2$ occur in a common V^{LR}_w. The *double cells* in $\hat{W}_\lambda$ are the $\approx_{LR}$ equivalence classes. Consider the multiset (set with multiplicities) $\{(\dim \sigma)\sigma \mid \sigma \in \hat{W}_\lambda\}$. A *PI* (for primitive ideal) *cell* in this multiset is a submultiset $\{m_\sigma \sigma\}$ such that $\sum m_\sigma \sigma$ is a left cell representation (or, equivalently, a right cell representation) of W_λ. Write $P(\mathscr{C}^L)$ for the cell attached to a left cell $\mathscr{C}^L$. Clearly,

$$\hat{W}_\lambda = \bigcup (\text{double cells in } \hat{W}_\lambda),$$

$$\{(\dim \sigma)\sigma\} = \bigcup_{\text{left cells } \mathscr{C}^L} P(\mathscr{C}^L).$$

LEMMA 2.13. *Suppose $m \geqslant 0$, $x \in \mathfrak{h}$. For $p = \sum_{y \in W_\lambda} c_y M(y\lambda) \in \mathbb{C}[W_\lambda]$, define*

$$r_m(p) = \sum c_y (y^{-1}x)^m \in S^m(\mathfrak{h}).$$

Then r_m is a W_λ-map from $\mathbb{C}[W_\lambda]$ with the right action, to the obvious W_λ action on $S(\mathfrak{h})$. The left action of W_λ on $\mathbb{C}[W_\lambda]$ corresponds to changing x in the definition of $r_m(p)$.

This is obvious.

DEFINITION 2.14. For $\sigma \in \hat{W}_\lambda$, define $a(\sigma)$ to be the smallest integer m such that σ occurs in $S^m(\mathfrak{h})$.

COROLLARY 2.15. *Suppose $I = I(w) \in \mathrm{Prim}_\lambda U(\mathfrak{g})$, and $\sigma = \sigma(I)$ is the Goldie rank representation of I (Definition 2.7). Then $a(\sigma) = a(w) = |\Delta^+| - Gk$ dimension $(L(w\lambda))$. If $\sigma' \leqslant_{LR} \sigma$ (Definition 2.12), then $a(\sigma') \leqslant a(\sigma)$. If $a(\sigma') = a(\sigma)$, then $\sigma' = \sigma$.*

This is a formal consequence of Theorem 2.6 and Lemma 2.11.

COROLLARY 2.16. *Each double cell in $\hat{W}_\lambda$ contains exactly one Goldie rank representation σ; it is characterized as having the minimum $a(\sigma)$ in the double cell. Each PI cell contains exactly one Goldie rank representation, and it has multiplicity one.*

Our terminology has been dangerously reminiscent of that used by Kazhdan and Lusztig in [17]. This cannot be completely justified. However, we will show that whenever the Kazhdan–Lusztig conjecture for the character formula for highest weight modules is valid, then the terminology is consistent. (The only gap is that we cannot show that every *PI* cell is a cell in Lusztig's sense [19].) This conjecture holds, for example, when $R_\lambda = \Delta(\mathfrak{g}, \mathfrak{h})$ (that is, when λ is integral) thanks to the work of Brylinski and Kashiwara and Beilinson and Bernstein [3, 7]. To avoid confusion in general, one must remember that the various definitions (of $\leqslant_L$ and so on) given here depend not only on the abstract Coxeter group W_λ, but also on $\mathfrak{g}$ and λ. Conjecturally this dependence is trivial, but this has not yet been proved completely. We will show, for example, that the double cells in $\hat{W}_\lambda$ depend only on the Coxeter group structure; but the question of determining $\leqslant_l$ on W_λ is much more subtle, and our techniques are probably not sufficient to treat it.

DEFINITION 2.17. Suppose $w \in W_\lambda$. Put

$$R_\lambda^+(w) = \{\alpha \in R_\lambda^+ \mid w^{-1}\alpha \notin R_\lambda^+\},$$

$$l(w) = |R_\lambda^+(w)|,$$

$$\tau_L(w) = R_\lambda^+(w) \cap \Pi_\lambda,$$

$$\tau_R(w) = \tau_l(w^{-1}),$$

the *left* and *right τ-invariants* of w (notation (2.2)). We often identify $\tau_L(w)$ or $\tau_R(w)$ with the corresponding subset of S_λ.

LEMMA 2.18. *Suppose $w \in W_\lambda$, $s \in S_\lambda$. Then*

$$\begin{aligned} l(ws) &= l(w) + 1, \qquad s \notin \tau_r(w), \\ &= l(w) - 1, \qquad s \in \tau_R(w). \end{aligned}$$

This is well known and very simple.

PROPOSITION 2.19 [6, 9]. *The right τ invariant $\tau_R(w)$ depends only on the primitive ideal $I(w)$; and the resulting map*

$$\tau\colon \operatorname{Prim}_\lambda U(\mathfrak{g}) \to \text{subsets of } S_\lambda \text{ or } \prod_\lambda$$

is an order-preserving homomorphism.

PROPOSITION 2.20 [12]. *Suppose $s \in S_\lambda$, and $w \in W_\lambda$.*

(a) *Suppose $s \in \tau_R(w)$. Then*

$$(1, s) \cdot L(w\lambda) = -L(w\lambda)$$

in $\mathbb{C}[W_\lambda]$ (Definition 2.8).

(b) *Suppose $s \notin \tau_R(w)$. Then*

$$(1, s) \cdot L(w\lambda) = L(w\lambda) + L(ws\lambda) + \sum_{\substack{y < w \\ s \in \tau_R(y)}} \mu_R^s(y, w) L(y\lambda).$$

Here $\mu_R^s(y, w)$ is a non-negative integer. Similar results hold on the left, with τ_R replaced by τ_L, and $(1, s)$ by $(s, 1)$.

It should be observed that the assertions on the left and right have very different proofs. If one uses the correspondence with characters of complex groups, however, the symmetry is recovered.

COROLLARY 2.21 (Duflo). *Suppose $w \in W_\lambda$, $s \in S_\lambda$, and $s \in \tau_L(w)$. Then $sw \leqslant_L w$.*

Unfortunately, the relations of Corollary 2.21 fail to generate $\leqslant_L$ (see [24]).

THEOREM 2.22 [17]. *There is a (combinatorially defined) function $\mu(y, w)$ on $W_\lambda \times W_\lambda$, with the following properties.*

(a) *$\mu(y, w) = \mu(w, y)$ is a non-negative integer; it is zero unless $l(y) - l(w)$ is odd.*

(b) *If w_0 is the longest element of W_λ, then*

$$\mu(y, w) = \mu(w_0 w, w_0 y) = \mu(ww_0, yw_0).$$

(c) *Suppose the (conjectural) Kazhdan–Lusztig character formulas hold for all $L(w\lambda)$, e.g., if λ is integral; see [7]. If $y \leqslant w \in W_\lambda$, $s \in S_\lambda$, $s \in \tau_R(y)$, and $s \notin \tau_R(w)$, then $\mu(y, w) = \mu_R^s(y, w)$, and $\mu(ws, w) = 1$.*

Without the Kazhdan–Lusztig conjecture, it is not even clear that $\mu_R^s(y, w) = \mu_R^{s'}(y, w) = \mu_L^{s''}(y, w)$ when all are defined.

COROLLARY 2.23. *Suppose the Kazhdan–Lusztig conjecture holds for all $L(w\lambda)$. Then the ordering $\leqslant_L$ (and related notation) defined above using primitive ideals, coincides with the corresponding ordering (also written $\leqslant_L$) defined in* [17].

This is just a combination of Theorem 2.22(c), Proposition 2.20(b), and Proposition 2.9(c). Since the result is included only to soothe the readers already familiar with the Kazhdan–Lusztig ordering, we will not repeat its definition here.

COROLLARY 2.24. *Suppose the Kazhdan–Lusztig conjecture holds for all $L(w\lambda)$. Then*

$$y \underset{L}{\leqslant} w \Leftrightarrow ww_0 \underset{L}{\leqslant} yw_0$$

$$\Leftrightarrow w_0 w \underset{L}{\leqslant} w_0 y,$$

and similarly for $\leqslant_R$, $\leqslant_{LR}$. In particular, there is a well-defined order-reversing involution of $\mathrm{Prim}_\lambda\, U(\mathfrak{g})$, $I \to I^{w_0}$; it is given by

$$I(w)^{w_0} = I(ww_0).$$

Lusztig has pointed out that this involution has a very simple description on the level of the Weyl group representations of Definition 2.12.

PROPOSITION 2.25 (Lusztig). *Suppose the Kazhdan–Lusztig conjecture holds for all $L(w\lambda)$. Then (notation* (2.12))

$$V_w^L \cong (V_{ww_0}^L)^* \otimes \mathrm{sgn},$$

as W_λ representation; and similarly for V^R, V^{LR}. Here sgn *denotes the sign character $w \to (-1)^{l(w)}$ of W_λ. The isomorphism is natural. There is a non-natural isomorphism*

$$V_w^L \cong V_{ww_0}^L \otimes \mathrm{sgn}.$$

[*Consequently, if $\{m_\sigma \sigma\}$ is a PI cell (Definition* 2.12), *then so is* $\{m_\sigma(\sigma \otimes \mathrm{sgn})\}$]. *In particular,* $\otimes$ sgn *defines a $\leqslant_{LR}$-reversing involution of $\hat{W}_\lambda$.*

Proof. V^L_w has a natural basis $\{L(z\lambda) \mid z \in \mathscr{C}^L_w\}$. By Theorem 2.22(c), an element $s \in S_\lambda$ acts by

$$\begin{aligned} s \cdot L(z\lambda) &= -L(z\lambda), s \in \tau_L(z\lambda) \\ &= L(z\lambda) + \sum_{\substack{y \in \mathscr{C}^L_w \\ s \in \tau_L(y)}} \mu(y, z), s \notin \tau_L(z\lambda). \end{aligned} \tag{2.26}$$

We give $(V^L_w)^* \otimes \operatorname{sgn}$ the dual basis $\tilde{D}(z)$, and set $D(z) = (-1)^{l(z)} \tilde{D}(z)$. Then s acts on this basis by the negative transpose of the matrix above, with the (y, z) entry multiplied by $(-1)^{l(y)+l(z)}$. By Theorem 2.22(a), this is

$$\begin{aligned} s \cdot D(z) &= L(z\lambda) + \sum_{\substack{y \in \mathscr{C}^L_w \\ s \notin \tau(y)}} \mu(y, z)\, L(y\lambda), s \in \tau_L(z) \\ &= -L(z\lambda), s \notin \tau_L(z). \end{aligned}$$

Send $D(z)$ to $L(zw_0\lambda)$. Since $(\mathscr{C}^L_w)\, w_0 = \mathscr{C}^L_{ww_0}$ by Corollary 2.24, this is a vector space isomorphism of $(V^L_w)^* \otimes \operatorname{sgn}$ onto $V^L_{ww_0}$. Since

$$\tau_L(zw_0) = S_\lambda - \tau_L(w_0),$$

comparison of the two formulas above with Theorem 2.22(b) shows that the isomorphism intertwines the action of s. Since S_λ generates W_λ, this proves the first assertion. Since every representation of W_λ is rational and therefore self-dual, the last claim follows. Q.E.D.

LEMMA 2.27. *Suppose W_λ has an exceptional factor, and $\mathfrak{g}$ is simple. Then the Kazhdan-Lusztig conjecture holds for all $L(w\lambda)$.*

We only sketch the proof, which is a straightforward application of the methods of [12]. By a reduction to a smaller algebra, we may assume R_λ spans $\mathfrak{h}$. If $R_\lambda = \Delta(\mathfrak{g}, \mathfrak{h})$, we are done by [7]. Otherwise, one checks easily that $\mathfrak{g}$ is of type E_8, and R_λ is $E_7 \oplus A_1$ or $E_6 \oplus A_2$. Suppose (which is the only difficult case) that $R_\lambda \cong E_6 \oplus A_2$, and that $w = (w_1, st)$ $(w_1 \in W(E_6), \{s, t\} = S_\lambda \cap W(A_2))$ lies in W_λ. It is easy to check that $L(wt\lambda) = L((w_1, s)\lambda)$ satisfies the conjecture, since (w_1, s) is in the Weyl group of the parabolic subgroup of type $E_6 \times A_1$; one reduces to the result in [7] for $E_6 \times A_1$. If $y < (w_1, s)$, then y is of the form (w_2, s) or $(w_2, 1)$, so $t \notin \tau_R(y)$. So

$$(1, t) \cdot L(wt\lambda) = L(wt\lambda) + L(w\lambda)$$

by Proposition 2.20(b). This computes the character of $L(w\lambda)$ explicitly.

Q.E.D.

When W_λ has no exceptional factors, we will rely on the results of [2]; and when there are such factors, the lemma allows us to apply Proposition 2.25.

PROPOSITION 2.28. *The double cell decompositions of $\hat{W}_\lambda$ and W_λ, the left cell decomposition of W_λ, and the notion of Goldie rank representation, depend only on W_λ as an abstract Coxeter group (and not on $\mathfrak{g}$ and λ).*

Proof. We may assume $\mathfrak{g}$ is simple. When all simple factors of W_λ are classical, the objects in question were explicitly described in [2], using only the Coxeter group structure. If W_λ has an exceptional factor, the result follows from Lemma 2.27. Q.E.D.

One would like also to have the corresponding result for the ordering $\leqslant_L$ on W_λ, corresponding to containments of primitive ideals.

THEOREM 2.29. *The equivalence relation $\approx_{LR}$ on $\hat{W}_\lambda$ is exactly the one described by Lusztig in* [19]: *that is, two representations are equivalent, if and only if they belong to cells in the sense of* [19] *with a common special representation. The Goldie rank representations are exactly the special representations in the sense of* [18] *or* [19].

For classical groups, this is contained in [2]; for exceptional groups, the proof will be given in Section 5. The equivalence relation is described in detail in [2] or [19] for the classical Weyl groups: two representations are equivalent exactly when their symbols in Lusztig's sense have the same underlying set. (This is not particularly easy to describe in clasical terms except for A_n, where the equivalence classes each have just one element.) For exceptional groups the equivalence classes are tabulated in [19]; in G_2, for example, there are three classes: the trivial representation, the sign representation, and everything else.

3. INDUCTION AND RESTRICTION

In this section, we study the relation between primitive ideals for $\mathfrak{g}$ and those for parabolic subalgebras. This will provide some basic information on the orderings of W_λ and $\hat{W}_\lambda$, which can then be greatly sharpened by applying Proposition 2.25. We begin with some standard results on the coset structure of W_λ. Fix a subset $S_\lambda^{\mathfrak{m}} \subseteq S_\lambda$ (notation (2.2)), or equivalently, $\Pi_\lambda^{\mathfrak{m}} \subseteq \Pi_\lambda$. Possibly after modifying our choice (2.3) of Δ^+, we may fix a parabolic $\mathfrak{p}$ in $\mathfrak{g}$ so that

$$\begin{gathered}\mathfrak{b} \subseteq \mathfrak{p} = \mathfrak{m} + \mathfrak{u} \qquad (\mathfrak{m} \supseteq \mathfrak{h} \text{ reductive, } \mathfrak{u} \subseteq \mathfrak{n} \text{ nilpotent}),\\ \Pi_\lambda \cap \Delta(\mathfrak{m}, \mathfrak{h}) = \Pi_\lambda^{\mathfrak{m}}.\end{gathered} \tag{3.1}$$

$$\begin{aligned}
R_\lambda^{\mathfrak{m}} &= R_\lambda \cap \Delta(\mathfrak{m}, \mathfrak{h}) = \text{span of } \Pi_\lambda^{\mathfrak{m}} \text{ in } R_\lambda,\\
W_\lambda^{\mathfrak{m}} &= W_\lambda \cap W(\mathfrak{m}, \mathfrak{h}) = W(R_\lambda^{\mathfrak{m}}),\\
M^{\mathfrak{m}}(\mu) &= U(\mathfrak{m}) \underset{\mathfrak{b}\cap\mathfrak{m}}{\otimes} \mathbb{C}_{\mu-\rho},\\
L^{\mathfrak{m}}(\mu) &= \text{irreducible quotient of } M^{\mathfrak{m}}(\mu),\\
I^{\mathfrak{m}}(\mu) &= \text{annihilator of } L^{\mathfrak{m}}(\mu) \text{ in } U(\mathfrak{m}),\\
\rho^{\mathfrak{m}} &= \tfrac{1}{2} \sum_{\alpha\in\Delta^+(\mathfrak{m},\mathfrak{h})} \alpha, \rho^{\mathfrak{u}} = \rho - \rho^{\mathfrak{m}},\\
I^{\mathfrak{m}}(w) &= \text{annihilator of } L^{\mathfrak{m}}(w\lambda) \qquad (w \in W_\lambda),\\
L^{\mathfrak{m}}(w\lambda) &= \sum_{y \in W_\lambda^{\mathfrak{m}} \cdot w} a_{y,w}^{\mathfrak{m}} M^{\mathfrak{m}}(y\lambda) \qquad (w \in W_\lambda).
\end{aligned} \tag{3.2}$$

The main thing to notice here is that the definition of $M^{\mathfrak{m}}(\mu)$ differs from that given for $\mathfrak{g}$: here we translate μ, not by half the sum of $\Delta^+(\mathfrak{m}, \mathfrak{h})$, but by half the sum of $\Delta^+(\mathfrak{g}, \mathfrak{h})$. This changes nothing serious (since $\rho - \rho^{\mathfrak{m}}$ is orthogonal to the roots of $\mathfrak{h}$ in $\mathfrak{m}$), and avoids introducing some shifts later.

The relations between $\mathfrak{m}$ and $\mathfrak{g}$ that we need come from

PROPOSITION 3.3 (see [12]). *Suppose $w \in W_\lambda$, and $y \in W_\lambda^{\mathfrak{m}} w$.*

(a) $a_{y,w}^{\mathfrak{m}} = a_{y,w}$.

(b) *$L(w\lambda)$ is a quotient of $U(\mathfrak{g}) \otimes_{\mathfrak{p}} L^{\mathfrak{m}}(w\lambda)$.*

(c) *Suppose $z \in W_\lambda^{\mathfrak{m}} w$ satisfies $\langle \alpha, z\lambda \rangle > 0$ for all $\alpha \in (R_\lambda^{\mathfrak{m}})^+$. Then*

$$a_{y,w}^{\mathfrak{m}} = a_{yz^{-1}, wz^{-1}}^{\mathfrak{m}}.$$

LEMMA 3.4. *The following conditions on an element $z \in W_\lambda$ are equivalent.*

(a) *For all $\alpha \in (R_\lambda^{\mathfrak{m}})^+$, $\langle \alpha, z\lambda \rangle > 0$.*

(b) $\tau_L(z) \cap \Pi_\lambda^{\mathfrak{m}} = \varnothing$.

(c) *$l(z)$ is minimal among all $\{l(w) \mid w \in W_\lambda^{\mathfrak{m}} z\}$.*

Write W_λ^L for the set of such z. Then the multiplication map

$$W_\lambda^{\mathfrak{m}} \times W_\lambda^L \to W_\lambda, (y, z) \to yz$$

is bijective.

This is elementary and well known.

DEFINITION 3.5. Let $\pi_L : W_\lambda \to W_\lambda^{\mathfrak{m}}$ be the map defined by $\pi_L(yz) = y$ ($y \in W_\lambda^{\mathfrak{m}}, z \in W_\lambda^L$). Similarly we define $\pi_R : W_\lambda \to W_\lambda^{\mathfrak{m}}$: thus

$$\pi_R(w) = \pi_L(w^{-1})^{-1}.$$

LEMMA 3.6. *Suppose $w \in W_\lambda$, and $y = \pi_L(w)$.*

(a) *There are finite dimensional representations F_1 and F_2 of $\mathfrak{m}$ so that $L^{\mathfrak{m}}(w\lambda)$ is a subquotient of $L^{\mathfrak{m}}(y\lambda) \otimes F_1$, and $L^{\mathfrak{m}}(y\lambda)$ of $L^{\mathfrak{m}}(w\lambda) \otimes F_2$.*

(b) *If $L^{\mathfrak{m}}(\mu)$ is a subquotient of $L(w\lambda)$ or of $U(\mathfrak{g}) \otimes_{\mathfrak{p}} L^{\mathfrak{m}}(w\lambda)$, (as an $\mathfrak{m}$ module), then there is a finite dimensional representation F of $\mathfrak{m}$ so that $L^{\mathfrak{m}}(\mu)$ occurs in $L^{\mathfrak{m}}(w\lambda) \otimes F$.*

(c) *$L^{\mathfrak{m}}(w\lambda)$ is a subquotient of $L(w\lambda)$.*

Proof. Part (a) is the translation principle for $\mathfrak{m}$ (which was already used implicitly, in the proof of Proposition 3.3(c)). For (b), Proposition 3.3(b) allows us to consider only $U(\mathfrak{g}) \otimes_{\mathfrak{p}} L^{\mathfrak{m}}(w\lambda)$. The claim follows, with $F = S^m(\mathfrak{g}/\mathfrak{p})$ for some m. Part (c) is also clear from this: in fact $L^{\mathfrak{m}}(w\lambda)$ is the $\mathfrak{m}$-submodule of $L(w\lambda)$ generated by the highest weight vector. Q.E.D.

COROLLARY 3.7. *The map $\pi_L : W_\lambda \to W_\lambda^{\mathfrak{m}}$ respects the ordering $\leqslant_R$; so π_R respects $\leqslant_L$. In particular,*

$$I(y) \subseteq I(w) \Rightarrow I^{\mathfrak{m}}(\pi_R(y)) \subseteq I^{\mathfrak{m}}(\pi_R(w)).$$

Thus π_R defines an order homomorphism from $\mathrm{Prim}_\lambda U(\mathfrak{g})$ *to* $\mathrm{Prim}_{\lambda - \rho(\mathfrak{u})} U(\mathfrak{m})$.

Proof. This follows from Proposition 2.9(d) and Lemma 3.6. Q.E.D.

LEMMA 3.8. *Let $<_{LC}$ be the preorder induced on $W_\lambda^L \cong W_\lambda^{\mathfrak{m}} \backslash W_\lambda$ by the Bruhat order $<$; that is, we require that if $w_1, w_2 \in W_\lambda$ and $z_1 \neq z_2 \in W_\lambda^L$, and $w_1 z_1 < w_2 z_2$, then $z_1 <_{LC} z_2$. Then $<_{LC}$ is an order (that is, there are no cycles).*

This is a standard fact about Coxeter groups; geometrically, it corresponds to the relation between the cell decompositions of G/B and G/P.

DEFINITION 3.9. Write $<_{LC}$ for the preimage on W_λ of the order $<_{LC}$ on $W_\lambda^L \cong W_\lambda^{\mathfrak{m}} \backslash W_\lambda$. That is, if $w_i \in W_\lambda^{\mathfrak{m}}$, $z_i \in W_\lambda^L$,

$$w_1 z_1 <_{LC} w_2 z_2 \Leftrightarrow z_1 <_{LC} z_2.$$

LEMMA 3.10. *Suppose* $y, w \in W_\lambda$, $s \in S_\lambda^{\mathfrak{m}}$, *and* $\mu_L^s(y, w)$ *is defined* (*Proposition* 2.20(b)).

(a) *If* $yw^{-1} \in W_\lambda^{\mathfrak{m}}$, *then*

$$\mu_L^s(y, w) = \mu_L^{\mathfrak{m}, s}(\pi_L(y), \pi_L(w))$$

(*the corresponding integer for* $\mathfrak{m}$).

(b) *If* $yw^{-1} \notin W_\lambda^{\mathfrak{m}}$, *and* $\mu_L^s(y, w) \neq 0$, *then* $y <_{LC} w$.

Proof. This follows from Propositions 2.20(b) and 3.3(c). Q.E.D.

PROPOSITION 3.11. *Suppose* $y, w \in W_\lambda$, $yw^{-1} \in W_\lambda^{\mathfrak{m}}$, *and* $\pi_L(y) \leqslant_L \pi_L(w)$. *Then* $y \leqslant_L w$. *In particular, if* $\mathscr{C}_w^L$ *is the left cell for* w (*Definition* 2.10), *and* $z \in W_\lambda^L$, *then*

$$\pi_L(\mathscr{C}_w^L \cap W_\lambda^{\mathfrak{m}} z) = \mathscr{C}_{x_1}^{\mathfrak{m}, L} \cup \cdots \cup \mathscr{C}_{x_t}^{\mathfrak{m}, L},$$

a union of left cells in $W_\lambda^{\mathfrak{m}}$. *So*

$$\pi_L(\mathscr{C}_w^L) = m_1 \mathscr{C}_{y_1}^{\mathfrak{m}, L} \cup \cdots \cup m_r \mathscr{C}_{y_r}^{\mathfrak{m}, L},$$

a union of left cells (*counted with the multiplicity with which they appear in the image of* π_L). *The corresponding left cell representations* (*Definition* 2.12) *satisfy*

$$V_w^L|_{W_\lambda^{\mathfrak{m}}} \cong \bigoplus_{i=1}^{r} m_i V_{y_i}^{\mathfrak{m}, L}.$$

Similar results hold with L replaced by R.

This result (which appears as Proposition 13 in [2]) is a consequence of Lemmas 3.10 and 3.8, and the formula (2.26) for V_w^L; the last isomorphism is obtained by filtering V_w^L using $<_{LC}$.

COROLLARY 3.12 (Joseph). *Suppose* $y, w \in W_\lambda$, $yw^{-1} \in W_\lambda^{\mathfrak{m}}$. *Then* $I^{\mathfrak{m}}(\pi_L(y)) \subseteq I^{\mathfrak{m}}(\pi_L(w)) \Rightarrow I(y) \subseteq I(w)$.

This should be compared with Corollary 3.7. If $S_\lambda^{\mathfrak{m}}$ consists of a single element, then Corollary 3.7 is Proposition 2.19, and Corollary 3.12 is Corollary 2.21.

We turn now to the problem of induction from $\mathfrak{m}$ to $\mathfrak{g}$. Fix $w \in W_\lambda^{\mathfrak{m}} \subseteq W_\lambda$, and consider the right cone $\overline{\mathscr{C}}_w^{\mathfrak{m}, R} \subseteq W_\lambda^{\mathfrak{m}}$ (Definition 2.10). We want to describe $\overline{\mathscr{C}}_w^R \subseteq W_\lambda$. By Proposition 3.3(a)

$$L(w\lambda) = U(\mathfrak{g}) \underset{\mathfrak{p}}{\otimes} L^{\mathfrak{m}}(w\lambda) = \sum_{y \leqslant w} a_{y, w}^{\mathfrak{m}} M(y\lambda).$$

So if $z \in W^L_\lambda$,

$$\begin{aligned}(1, z)^{-1}L(w\lambda) &= \sum_{y \in W_\lambda} a^{\mathfrak{m}}_{y,w} M(yz\lambda)\\ &= U(\mathfrak{g}) \underset{\mathfrak{p}}{\otimes} L^{\mathfrak{m}}(wzy),\end{aligned}$$

which contains $L(wz\lambda)$ as a subquotient by Proposition 3.3(b). By Proposition 2.9(b) and Definition 2.10, $w \leqslant_R wz$. Suppose $y \in W^{\mathfrak{m}}_\lambda$, and $w \leqslant_R y$. By the argument just given, $y \leqslant_R yz$; so $w \leqslant_R yz$. This shows that if $x \in W_\lambda$, then,

$$\pi_L(x) \underset{R}{\geqslant} W \Rightarrow x \underset{R}{\geqslant} w.$$

Corollary 3.7 is the converse, so we find

$$\begin{aligned}\bar{\mathscr{C}}^R_w &= \{x \in W_\lambda \mid \pi_L(x) \underset{R}{\geqslant} w\}\\ &= (\bar{\mathscr{C}}^{\mathfrak{m},R}_w)\, W^L_\lambda \qquad (w \in W^{\mathfrak{m}}_\lambda).\end{aligned} \tag{3.13}$$

By Lemma 3.6(b), any composition factor of $U(\mathfrak{g}) \otimes_{\mathfrak{p}} L^{\mathfrak{m}}(x\lambda)$ is of the form $L(y\lambda)$, with $\pi_L(y) \geqslant_R \pi_L(x)$. By a simple induction, we conclude that

$$\{U(\mathfrak{g}) \underset{\mathfrak{p}}{\otimes} L^{\mathfrak{m}}(x\lambda) \mid x \in \bar{\mathscr{C}}^R_w\} \text{ is a basis of } \bar{V}^R_w. \tag{3.14}$$

The right action of W_λ is easy to compute in this basis; so we find:

Proposition 3.15. *Suppose $w \in W^{\mathfrak{m}}_\lambda$. Then*

(a) $\bar{\mathscr{C}}^R_w = \bar{\mathscr{C}}^{\mathfrak{m},R}_w W^L_\lambda$.

(b) $\bar{V}^R_w \cong \operatorname{Ind}^{W_\lambda}_{W^{\mathfrak{m}}_\lambda}(\bar{V}^{\mathfrak{m},R}_w)$.

Similar statements hold with R replaced by L.

For the last statement, one uses the fact that L and R can always be interchanged, using the interpretation in terms of Harish-Chandra modules for a complex group. The proof given here of the first statements is *not* symmetric in L and R.

Definition 3.16. Suppose $w \in W^{\mathfrak{m}}_\lambda$. The cone $\bar{\mathscr{C}}^R_w$ is called an *induced (right) cone* from $\mathfrak{m}$ to $\mathfrak{g}$, because of Proposition 3.15. The cell $\mathscr{C}^R_w$ is called an *induced (right) cell.*

Proposition 3.15 is a complete description of induced cones; it allows one to pass from information about $\leqslant_{L,R}$ for $\hat{W}^{\mathfrak{m}}_\lambda$ to similar information for $\hat{W}_\lambda$, using only ordinary induction for finite groups. The situation for cells is not so satisfactory. We begin by explaining (following Lusztig) what the answer should be like, and then give some *ad hoc* techniques which make it possible to do calculations.

DEFINITION 3.17. Suppose $\sigma \in \hat{W}_\lambda$. Let σ_0 be the unique Goldie rank representation in the same double cell as σ (cf. Corollary 2.16). Put

$$\tilde{a}_{PI}(\sigma) = a(\sigma_0)$$

(notation (2.14)).

By Corollary 2.16, $\tilde{a}_{PI}(\sigma) \leqslant a(\sigma)$, with equality exactly when σ is a Goldie rank representation.

PROPOSITION 3.18. *Suppose $w \in \hat{W}_\lambda^{\mathfrak{m}}$. Write $V_w^{\mathfrak{m},L}$ for the left cell representation, and $\sigma_0^{\mathfrak{m}}$ for the unique Goldie rank representation of* $\mathfrak{m}$ *occurring in it. Similarly define V_w^L and σ_0 for W_λ (using w).*

(a) $a(w) = a_{(w)}^{\mathfrak{m}}$ (*notation as in Theorem* 2.6).

(b) $a(\sigma_0) = a^{\mathfrak{m}}(\sigma_0^{\mathfrak{m}})$ (*notation* (2.14)).

(c) $$V_w^L \cong \sum_{\substack{\sigma \in \hat{W}_\lambda \\ \tilde{a}_{PI}(\sigma) = \tilde{a}_{PI}^{\mathfrak{m}}(\sigma_0^{\mathfrak{m}})}} (\dim \operatorname{Hom}_{W_\lambda^{\mathfrak{m}}}(\sigma|_{W_\lambda^{\mathfrak{m}}}, V_w^{\mathfrak{m},L}))\sigma.$$

as a representation of W_λ.

Proof. Theorem 2.6(b) shows how to compute $a(w)$ from the character of $L(w\lambda)$; so (a) follows from Proposition 3.3(a). Part (b) is Corollary 2.15 and (a). For (c), we know that V_w^L is obtained from $\bar{V}_w^L$ by removing various V_y^L with $I(y) \supsetneqq I(w)$. If $I(y) \supseteq I(w)$, then $I(y) \neq I(w)$ if and only if

$$GK \dim L(y\lambda) < GK \dim L(w\lambda);$$

that is, $a(y) > a(w)$. Now if σ occurs in V_y^L, then

$$\tilde{a}_{PI}(\sigma) = a(y); \tag{3.19}$$

this is substantially the definition (by Corollary 2.16). So (c) is a consequence of Proposition 3.15(b). Q.E.D.

This proposition is pretty but not decisive; one of our aims is to compute the double cells in $\hat{W}_\lambda$, and $\tilde{a}_{PI}(\sigma)$ is defined in terms of these unknown cells. We will state now the answer (confirming a conjectire of Lusztig). The proof amounts to a case-by-case computation of both side separately. To formulate the result, we need to recall a definition from [18]; we refer to that paper for details. Let W be a Weyl group; and for each prime power q, let $G(q)$ be a split Chevalley group over $\mathbf{F}_q$ with Weyl group W. Let $B(q)$ be a Borel subgroup of $G(q)$. To each $\sigma \in \hat{W}$, one can associate canonically a complex representation $\pi(\sigma) \in \hat{G}(q)$, having a $B(q)$-fixed vector. There is a polynomial $\tilde{p}_\sigma$ (independent of q) such that

$$\dim \pi(\sigma) = \tilde{p}_\sigma(q).$$

Then $\tilde{a}(\sigma)$ is defined to be the degree of the lowest degree term in p_σ. (Since $G(q)$ is not canonically associated to W, this is not obviously well defined; but in fact it is.) These numbers (and even the polynomials $\tilde{p}_\sigma$) are explicitly computed in all cases (see [18]).

THEOREM 3.20. *For any $\sigma \in \hat{W}_\lambda$, $\tilde{a}_{PI}(\sigma) = \tilde{a}(\sigma)$ (notation (3.17) and [18]).*

This follows from Theorem 2.29 (if one keeps in mind the explicit values of $\tilde{a}(\sigma)$ given in [18]).

COROLLARY 3.21. *Suppose the Kazhdan–Lusztig character formulas hold for all $L(w\lambda)$. Then every cell in the sense of Lusztig [19] is a PI cell.*

Proof. Lusztig's cells (by definition) are obtained either by induction from a parabolic as in Proposition 3.18(c), with $\tilde{a}$ replacing $\tilde{a}_{PI}$; or by tensoring such a cell with the sign representation of W_λ. Both operations preserve *PI* cells (in the latter case by Proposition 2.25). Q.E.D.

We turn now to methods for computing $\tilde{a}_{PI}$.

LEMMA 3.22. *The map $\sigma \to \tilde{a}_{PI}(\sigma)$, from $(\hat{W}_\lambda, \leqslant_{LR})$ to $(N, \leqslant)$ is order preserving. If $\sigma_1 \leqslant_{LR} \sigma_2$ and $\sigma_1 \not\approx_{LR} \sigma_2$, then $\tilde{a}_{PI}(\sigma_1) < \tilde{a}_{PI}(\sigma_2)$.*

This is a consequence of Corollary 2.15 and the definitions.

LEMMA 3.23. *Suppose $\sigma_0^{\mathfrak{m}} \in \hat{W}_\lambda^{\mathfrak{m}}$ is a Goldie rank representation, $\sigma_1^{\mathfrak{m}} \in \hat{W}_\lambda^{\mathfrak{m}}$, and $\sigma_1^{\mathfrak{m}} \geqslant_{LR} \sigma_0^{\mathfrak{m}}$. Fix $\sigma \in \hat{W}_\lambda$.*

(a) *$\tilde{a}_{PI}(\sigma) \leqslant a(\sigma)$, with equality exactly when σ is a Goldie rank representation.*

(b) *If σ occurs in $\mathrm{Ind}_{W_\lambda^{\mathfrak{m}}}^{W_\lambda}(\sigma_1)$, then $\tilde{a}_{PI}(\sigma) \geqslant \tilde{a}_{PI}^{\mathfrak{m}}(\sigma_1^{\mathfrak{m}}) \geqslant \tilde{a}_{PI}^{\mathfrak{m}}(\sigma_0^{\mathfrak{m}}) = a^{\mathfrak{m}}(\sigma_0^{\mathfrak{m}})$.*

(c) *For fixed $\sigma_0^{\mathfrak{m}}$, there is exactly one $\sigma_0 \in \hat{W}_\lambda$ which occurs in $\mathrm{Ind}(\sigma_0^{\mathfrak{m}})$ and satisfies $a(\sigma_0) = a^{\mathfrak{m}}(\sigma_0^{\mathfrak{m}})$. This σ_0 is a Goldie rank representation.*

Proof. Part (a) has already been observed (after Definition 3.17). Parts (b) and (c) are contained in Propositions 3.18 and 3.15, and Lemma 3.22. Q.E.D.

DEFINITION 3.24. Suppose $\sigma_0^{\mathrm{m}} \in \hat{W}_\lambda^{\mathrm{m}}$ occurs exactly once (and not more often) in degree $a^{\mathrm{m}}(\sigma_0^{\mathrm{m}})$ in $S(\mathfrak{h})$. (This will be the case if σ_0^{m} is a Goldie rank representation, by Theorem 2.6(d)). Let σ_0 be the unique element of $\hat{W}_\lambda$ which occurs in $\mathrm{Ind}_{W_\lambda^{\mathrm{m}}}^{W_\lambda}(\sigma_0^{\mathrm{m}})$, and satisfies $a(\sigma_0) = a^{\mathrm{m}}(\sigma_0^{\mathrm{m}})$. (This exists by [20]; for Goldie rank representations it is Lemma 3.23(c).) Write

$$\sigma_0 = j_{W_\lambda^{\mathrm{m}}}^{W_\lambda}(\sigma_0^{\mathrm{m}}) = j_{\mathrm{m}}(\sigma_0^{\mathrm{m}}) = j(\sigma_0^{\mathrm{m}});$$

we call j *truncated induction.*

The condition imposed on σ_0^{m} is empty except in types D_{2n}, E_7, and E_8 [5]. For type A_{n-1}, truncated induction is very familiar: if W_λ^{m} corresponds to a partition $\xi = (\xi_1 \geqslant \xi_2 \geqslant \cdots)$ of n (that is, if W_λ^{m} is of type $A_{\xi_1 - 1} \oplus \cdots)$ then

$$j(\mathrm{sgn}^{\mathrm{m}}) = \sigma_\xi,$$

the representation of the symmetric group attached to the partition ξ.

COROLLARY 3.25 (Joseph). *The class of Goldie rank representations of Weyl groups is closed under truncated induction.*

To make sense of this, we need to recall Proposition 2.28.

4. ORBITAL INTEGRALS

Let $\mathcal{N}$ denote the nilpotent cone in $\mathfrak{g}^*$, the dual of the Lie algebra $\mathfrak{g}$. (We may speak of nilpotent elements in $\mathfrak{g}^*$ by identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ using an invariant bilinear form.) Fourier inversion of nilpotent orbital integrals on the Lie algebra is equivalent to the corresponding problem on the group, so we consider only the former. Recall Harish-Chandra's invariant integral

$$C_c^\infty(\mathfrak{g}^*) \to C_c^\infty(\mathfrak{h}^*)^W, f \to \phi_f \tag{4.1}$$

[25, Chap. 8]). The complexification of $\mathfrak{h}^*$ regarded as a real vector space is $\mathfrak{h}^* \oplus \mathfrak{h}^*$; so the real linear constant coefficient differential operators on $\mathfrak{h}^*$ may be identified with $S(\mathfrak{h}^* \oplus \mathfrak{h}^*) = S(\mathfrak{h}^*) \otimes S(\mathfrak{h}^*)$.

DEFINITION 4.2. Suppose $\sigma_0 \in \hat{W}_\lambda$ occurs exactly once in degree $a(\sigma_0)$ in $S(\mathfrak{h}^*)$ (Definition 2.14). Let $\sigma \in \hat{W}$ be the (irreducible) representation of W it generates there, and $V_\sigma \subseteq S(\mathfrak{h}^*)$ the corresponding subspace. Let $L_{\sigma_0} = L_\sigma \subseteq V_\sigma \otimes V_\sigma \subseteq S(\mathfrak{h}^*) \otimes S(\mathfrak{h}^*)$ be the unique line on which the diagonal Weyl group acts trivially. Fix some non-zero element $D_{\sigma_0} = D_\sigma$ of L_{σ_0}; we regard D_{σ_0} as a differential operator on $\mathfrak{h}^*$ with constant coefficients as above. D_{σ_0} is called the *differential operator attached to* σ_0 (or σ).

DEFINITION 4.3. A representation $\sigma \in \hat{W}$ is called a *generalized Goldie rank representation* if there is a regular $\lambda \in \mathfrak{h}^*$ and a Goldie rank representation $\sigma_0 \in \hat{W}_\lambda$, such that σ is obtained from σ_0 as in Definition 4.2.

Since the set of possible W_λ depends on the root system, and not just on W, the set of generalized Goldie rank representations does as well. Since we know all Goldie rank representations, and all possible W_λ can be determined, we know all generalized Goldie rank representations. In fact (by Theorem 2.29), they have already been studied by Lusztig; and we have

COROLLARY 4.4 (of Theorem 2.29). *The set of generalized Goldie rank representations is the set denoted* $\bar{\mathscr{S}}_W$ *in* [18], *which we might call generalized special representations.*

Springer [22] has defined an injective correspondence from orbits in $\mathscr{N}$ (that is, nilpotent conjugacy classes) to representations of W. Lusztig and Alvis [1], using work of several people, have recently completed the explicit calculation of this correspondence, in terms to the known classifications of both sides. (For classical groups this may be found in [18]). In particular, they prove

THEOREM (4.5 (Alvis and Lusztig [1]). *The image of the Springer correspondence is exactly* $\bar{\mathscr{S}}_W$.

This suggests the following conjecture.

Conjecture 4.6. Let $\mathcal{O}$ be an orbit in $\mathscr{N}$, $\sigma \in \bar{\mathscr{S}}_W \subseteq \hat{W}$ the corresponding representation, and D_σ the associated differential operators (Definition 4.2). Then the distribution

$$f \to [D_\sigma(\phi_f)](0)$$

(notation (4.1)) is an invariant measure on $\mathcal{O}$.

This is not quite as unfounded as it might first appear. It is quite easy to see that the distribution in question is supported on $\mathscr{N}$, and has the same homogeneity degree as the invariant measure. Furthermore, Harish-Chandra has shown that the invariant measure is of this general form, possibly with some other differential operator replacing D_σ. When $\mathcal{O} = \{0\}$ (so that σ is the sign representation, and D_σ is an appropriate product of the roots), the formula

$$f(0) = (D_{\mathrm{sgn}}(\phi_f))(0)$$

is the main step in the proof of the Plancherel theorem for complex reductive groups. (An identical argument computes the Fourier transform of $\mathcal{O}$ from D_σ in general, given the conjecture. We will therefore say no more about Fourier inversion as such.) Our main result on orbital integrals is

THEOREM 4.7. *Conjecture* 4.6 *is true whenever* $\mathcal{O}$ *is a special nilpotent orbit in a classical algebra. When* $\mathfrak{g}$ *is an exceptional algebra, Conjecture* 4.6 *is always true, except perhaps in* E_8 *for the two orbits of dimension* 202 *and the non-induced orbit of dimension* 188. *These have Dynkin diagrams*

$$\begin{array}{ccc} 1\;0\;1\;0\;0\;1\;0 & \quad 1\;0\;0\;1\;0\;1 \quad & 0\;0\;0\;1\;0\;0\;1 \\ 0 & 0 & 0 \end{array}$$

The first assertion is established in [2], and the second will be proved in Section 5. We now recall how this theorem is related to the study of primitive ideals.

THEOREM 4.8 [2]. *To each* $w \in W_\lambda$, *we can associate a closed Ad-invariant subset* $\overline{\mathcal{O}}_w \subseteq \mathcal{N}$, *of dimension* $2(|\Delta^+| - a(w))$ (*cf. Theorem* 2.6). *Write*

$$\mathcal{O}_w = \mathcal{O}_w^1 \cup \cdots \cup \mathcal{O}_w^r$$

for the orbits of dimension $2(|\Delta^+| - a(w))$ *in* $\overline{\mathcal{O}}_w$. *Then* $\overline{\mathcal{O}}_w$ *carries a natural Ad-invariant signed measure* μ_w; *and* μ_w *is positive on at least one of the* $\mathcal{O}_w^i$. *If* $\sigma(w)$ *is defined by Definition* 2.7, *then we have for* $f \in C_c^\infty(\mathfrak{g}^*)$,

$$\int_{\overline{\mathcal{O}}_w} f\, d\mu_w = [D_{\sigma(w)}\, \phi_f](0)$$

(*Defintion* 4.2), *for an appropriate normalization of* $D_{\sigma(w)}$. *The set* $\overline{\mathcal{O}}_w$ *depends only on the double cell of* w *in* W_λ (*Definition* 2.10); *and, up to positive scalar multiples, the same in true for* μ_w. *The map* $w \to \overline{\mathcal{O}}_w$ *is order reversing for* $\leqslant_{LR}$ (*Defintion* 2.10). *Finally, write* $\mathrm{gr}(I(w))$ *for the associated graded ideal in* $S(\mathfrak{g})$, *and* $\mathcal{V}(\mathrm{gr}(I(w)))$ *for its associated variety in the maximal spectrum* $\mathfrak{g}^*$ *of* $S(\mathfrak{g})$. *Then*

$$\overline{\mathcal{O}}_w \subseteq \mathcal{V}(\mathrm{gr}(I(w))) \subseteq \mathcal{N}$$

and the first two terms have the same dimension.

This is proved by looking at representations of a complex group with Lie algebra $\mathfrak{g}$; $\overline{\mathcal{O}}_w$ is Howe's wavefront set [11] of a representation naturally attached to w and λ, and the statements about μ_w come from study of the behavior of its character near the identity.

COROLLARY 4.9. *Suppose* $\sigma \in \mathscr{S}_W \subseteq \hat{W}$. *Then the distribution*

$$\Omega_\sigma = \{f \to [D_\sigma \phi_f](0)\}$$

is a linear combination of invariant measures on hilpotent orbits. Conversely, the invariant measure on any nilpotent orbit $\mathcal{O}$ *is a linear combination of the various* Ω_σ, *with* $\sigma \in \mathscr{S}_W$, *and* $a(\sigma) = \frac{1}{2}(\dim \mathfrak{g}/\mathfrak{h} - \dim \mathcal{O})$.

Proof. The first assertion is Theorem 4.8 and Corollary 4.4. For the second, Theorem 4.5 implies that there are exactly $|\mathscr{S}_W|$ nilpotent orbits. (Of course, this fact is much more elementary than the full result of Theorem 4.5). So it is enough to observe that the various Ω_σ are linearly independent distributions. This is easy to check, for example, by computing their Fourier transforms. The statement about $a(\sigma)$ follows by inspection of homogeneity degrees. Q.E.D.

Because of Theorem 4.8, the following conjecture would imply Conjecture 4.6.

Conjecture 4.10 (Borho, Jantzen, Joseph). $\mathscr{V}(\mathrm{gr}(I(w)))$ is the closure of the orbit corresponding to $\sigma(w)$ in the Springer correspondence.

This has been completely proved only for $\mathfrak{sl}(n)$ and small $\mathfrak{g}$, and seems to be quite difficult. Therefore we will confine our attention to $\mathscr{T}_w$, which is less natural, but easier to compute. The most pleasant feature of $\mathscr{T}_w$ is that it behaves well under induction.

PROPOSITION 4.11 [2]. *In the setting of* (3.1) *and* (3.2), *suppose* $w \in W_\lambda^{\mathfrak{m}}$; *define* $\mathscr{T}_w^{\mathfrak{m}}$ *by Theorem* 4.8 *for* $\mathfrak{m}$. *Then*

$$\mathscr{T}_w \subseteq \mathrm{Ad}(G) \cdot (\mathscr{T}_w^{\mathfrak{m}} + \mathfrak{u}).$$

Here $H = \mathrm{Ad}(\mathfrak{g})$, *and* $\mathfrak{u}$ *is the nil radical of the parabolic subalgebra under consideration.*

5. THE EXCEPTIONAL ALGEBRAS

We have to prove Theorems 2.29 and 4.7 when $\mathfrak{g}$ is an exceptional simple Lie algebra. We begin with Theorem 2.29. By Proposition 2.28, we may assume λ is integral; so $W_\lambda = W$. We will be a little sketchy, since the proofs consist mostly of consulting tables. More details are provided for F_4 in an appendix, as an example.

Step I. If two representations σ_1, σ_2 *of* W *lie in a common cell in the sense of* [19], *then* $\sigma_1 \approx_{LR} \sigma_2$.

This is proved case by case and representation by representation, starting with the smaller groups and building up. At the same time, we keep track of the structure of some left cells as well as two-sided cells. More precisely, the inductive hypothesis we need is that if $\{\delta_1, \delta_2, \delta_3\}$ is a Lusztig equivalence class in $\hat{W}$ with exactly three elements, then any left cell containing δ_1 contains exactly one of $\{\delta_2, \delta_3\}$. So suppose first that Lusztig's equivalence class for σ_1 and σ_2 is $\{\sigma_1, \sigma_2, \sigma_3\}$, with σ_1 special. By Proposition 2.25, we may, if necessary, replace σ_1, σ_2 by $\sigma_1 \otimes \text{sgn}$, $\sigma_2 \otimes \text{sgn}$. Then Theorem 6 of [18] assures that we can find a maximal proper parabolic $W^{\mathfrak{m}} \subseteq W$ (notation as in Section 3) and a special representation $\delta_1 \in \hat{W}^{\mathfrak{m}}$, with $\sigma_1 = j(\delta_1)$ (Definition 3.24). Case by case, one sees that δ_1 may be chosen in a three element double cell $\{\delta_1, \delta_2, \delta_3\}$, with δ_i occurring in $\sigma_i|_{W^{\mathfrak{m}}}$. Since $\sigma_1|_{W^{\mathfrak{m}}}$ contains δ_1 exactly once, Proposition 3.11 assures that if $\sigma_1 + \sum m_\sigma \sigma$ is a left cell representation, then $(\sigma_1 + \sum m_\sigma \sigma)|_{W^{\mathfrak{m}}}$ contains exactly one of δ_2 and δ_3, exactly once; and each possibility occurs for some left cell containing σ_1. By Lemma 3.23 (a), $a(\sigma) < a(\sigma_1)$ whenever $m_\sigma \neq 0$. Using Alvis's tables [26] for restricting to $W^{\mathfrak{m}}$, one can check that this forces each left cell containing σ_1 to contain exactly one of σ_2 and σ_3, exactly once. (This seems to be something for nothing. The point is that the induction pushes the problem down into classical subalgebras. There this argument fails for three element double cells in B_2 and D_4; and one needs to do some work, for example, as in the remaining cases below, to prove the claim.)

So we may assume that Lusztig's equivalence class for σ has two, or more than three, elements. For G_2, the primitive ideals are described in [6], and the desired result follows easily. In F_4 and E_6, there is exactly one special representation σ such that σ is isomorphic to $\sigma \otimes \text{sgn}$, and its equivalence class is the only one having either two or more than three elements. We may as well assume $\sigma = \sigma_1$. Using Proposition 3.15 and Alvis's tables, one checks in each case that $\sigma_1 \leqslant_{LR} \sigma_2$. (That is, one writes $\sigma_1 = j(\delta_1)$, and finds $\delta_2 \geqslant_{LR} \delta$ so that δ_2 occurs in $\sigma_2|_{W_{\mathfrak{m}}}$.) Since $\sigma_2 \otimes \text{sgn}$ is also in Lusztig's equivalence class for σ_1, we have also $\sigma_1 \leqslant_{LR} \sigma_2 \otimes \text{sgn}$. By Proposition 2.25 and the hypothesis $\sigma_1 \cong \sigma_1 \otimes \text{sgn}$, we conclude the $\sigma_1 \geqslant_{LR} \sigma_2$. So $\sigma_1 \approx_{LR} \sigma_2$. Exactly the same argument applies to the equivalence class of the Weyl group representation 4480_y in E_8. In E_7, we have to consider the classes of $512_a'$, $315_a'$, and 315_a. For the first, Lusztig's class is $\{512_a', 512_a\}$. By Proposition 3.15 applied to the E_6 parabolic, one computes $512_a \geqslant_{LR} 512_a'$. Tensoring with sgn interchanges these two representations, so they are $\approx_{LR}$ equivalent. For $315_a'$, we use the E_6 parabolic; then $315_a' = j(80_s)$. So if $\delta \in W(\hat{E}_6)$ and $\delta \approx_{LR} 80_s$, we can find a $\sigma \in \hat{W}$, with $\sigma \approx_{LR} 315_a'$, so that δ occurs in $\sigma|_{W(E_6)}$. (This is a consequence of Proposition 3.11.) By inspection of Alvis's tables, this forces everything in Lusztig's equivalence class to be $\approx_{LR}$ equivalent to $315_a'$. Tensoring with sign gives the result for 315_a. In E_8, we must consider the classes of 1400_z, 1400_x, 4096_z, and their tensor products with sgn. (The remaining bad class, 4480_y, has already been considered.) Consider first 1400_x and 1400_z; and fix some σ_0 in the double cell of $315'_a$ in E_7. Using Alvis's tables, one lists all representations for E_8 which contain σ_0, and eliminates all those which we already know to be in the same double cell as a Goldie rank representation other than 1400_x and 1400_z. There will be just two left, say $\{\sigma_1, \sigma_2\}$; and this sets up a one-to-two correspondence between the double cell of $315'_a$, and the union of Lusztig's equivalence classes of 1400_x and 1400_z. For (say) σ_1, we can prove $\sigma_1 \geqslant_{LR} 1400_x$ by Proposition 3.15; and since $1400_x \geqslant_{LR} 1400_z$ by the same proposition, it follows that $\sigma_1 \not\leqslant_{LR} 1400_z$. On the other hand, Proposition 3.11 assures that each of 1400_x and 1400_z is in the same double cell as one of σ_1 and σ_2; so $\sigma_1 \approx_{LR} 1400_x$, $\sigma_2 \approx_{LR} 1400_z$. Tensoring with sgn treats $1400'_x$ and $1400'_z$. For 4096_z and $4096'_x$, one uses Proposition 3.15 to show that $4096_z \leqslant_{LR} 4096_x$, and $4096'_x \leqslant_{LR} 4096'_z$. By Proposition 2.25, the reverse inequalities hold as well.

Step II. If σ_1 and σ_2 are distinct special representations of W, then $\sigma_1 \not\leqslant_{LR} \sigma_2$.

Again we work case by case by induction on dim $\mathfrak{g}$. If σ_1 and σ_2 are both obtained by truncated induction (from special representations) then they are both Goldie rank representations, and we are done. If $\sigma_1 \otimes$ sgn and $\sigma_2 \otimes$ sgn are both obtained by truncated induction, then $\sigma_1 \otimes \text{sgn} \not\leqslant_{LR} \sigma_2 \otimes \text{sgn}$ by the first case, so $\sigma_1 \not\leqslant_{LR} \sigma_2$ by Proposition 2.25. Next, suppose there is a special $\sigma_3 \neq \sigma_1$ such that σ_1 and σ_3 are both obtained by truncated induction; and

$$\sigma_1 \leqslant_{LR} \sigma_3 \leqslant_{LR} \sigma_2.$$

Then $\sigma_1 \not\leqslant_{LR} \sigma_3$ by the first case; so $\sigma_1 \not\leqslant_{LR} \sigma_2$ *a fortiori*. Since most special representations are obtained by truncated induction, it is easy to verify (using Alvis's tables) that we are always in one of these three cases. This completes the proof of Theorem 2.29. Q.E.D.

We turn now to the proof of Theorem 4.7. The main tools are Theorem 4.8 and Proposition 4.11; and we proceed case by case, by inspection of Dynkin's tables in [10]. By Proposition 4.11, we only have to consider the non-induced nilpotents (in the sense of [20]). By Corollary 4.9, we can also eliminate those cases in which there is only one nilpotent orbit of a certain dimension. This leaves only four orbits in E_8: those omitted by the theorem, and one of dimension 200 and diagram

$$\begin{matrix} 0 \\ 0100100 \end{matrix}.$$

The corresponding Springer representation is 420_y. This arises as a Goldie rank representation $\sigma(w_0)$ for W_A of type $A_4 \times A_4$, and w_0 the long element of W_A. Now let w_1 be the long element of an $A_3 \times A_4$ subsystem in W_A.

Then $w_0 \geqslant_{LR} w_1$, so $\bar{\mathscr{N}}_{w_0} \subseteq \bar{\mathscr{N}}_{w_1}$. By Proposition 4.11, $\bar{\mathscr{N}}_{w_1}$ is the closure of a certain 208 dimensional Richardson orbit. By calculations of Spaltenstein [21], it follows that $\bar{\mathscr{N}}_{w_1}$ does not contain the other 200 dimensional nilpotent orbit; so $\mathscr{N}_{w_0}$ must be exactly the orbit we want. The result now follows from Theorem 4.8. Q.E.D.

To treat the remaining three orbits, this idea is definitely inadequate: they cannot be distinguished by knowledge of which closures of larger orbits contain them.

6. Open Problems

The discussion in Section 4 of the connections among nilpotent orbits, primitive ideals, and the Springer correspondence, raises many more questions. One hint of what else is going on is this.

Conjecture 6.1. Fix $w \in W_\lambda$, and write

$$V = (\operatorname{Ind}_{W_\lambda}^{W} \bar{V}_w^L) \otimes \operatorname{sgn}$$

(notation (2.10)). Then, as a representation of W, the coordinate ring of the scheme-theoretic intersection $\bar{\mathscr{N}}_w \cap \mathfrak{h}$ contains V. By [8], this is true for type A; and in fact equality holds. For other types, equality cannot hold since V does not depend only on $\bar{\mathscr{N}}_w$. Possibly the coordinate ring is some sort of maximum of all possible V.

Obviously one would like a detailed knowledge of the actual fibers of the map $w \to I(w)$; that is, of the left cells in W_λ. It is not clear how to formulate a conjecture along these lines. As a start, however, one might try to compute the left cells as Weyl group representations. This is done by Proposition 3.18 for induced cells, and Proposition 2.25 does it for the opposite kind of cell. Lusztig has suggested that all left cells are of one of these two forms, up to isomorphism. For supporting evidence, we have:

Proposition 6.2. *Suppose α, β are adjacent roots in $\prod_\lambda$ [cf. (2.2)] spanning an A_2. Fix $w \in W_\lambda$, and suppose $\alpha \in \tau_R(w)$, $\beta \notin \tau_R(w)$. Then $\mathscr{C}_w^L$ (Definition 2.10) is contained in the domain of $T_{\alpha\beta}$ [24]; and $T_{\alpha\beta}$ defines a bijection*

$$T_{\alpha\beta} : \mathscr{C}_w^L \to \mathscr{C}_{T_{\alpha\beta}(w)}^L.$$

The induced isomorphism $V_w^L \to V_{T_{\alpha\beta}(w)}^L$ respect the action of W_λ. If α and β span a B_2, identical results hold with $T_{\alpha\beta}$ replaced by $S_{\alpha\beta}$ [24].

This is easy to check; except for the W_λ action, it is contained in [24] and the references there.

Conjecture 6.3 (Lusztig). Every left cell is isomorphic by a sequence of $T_{\alpha\beta}$'s (with α and β spanning an A_2) or $S_{\alpha\beta}$'s to an induced left cell; or to a left cell opposite to an induced one in the sense of Proposition 2.25.

This would imply Lusztig's conjecture that the "cells" of [19] are exactly the left cells as Weyl group representations. One would also like to know how many left cells are in each of these equivalence classes.

The Duflo map $W_\lambda \to \mathrm{Prim}_\lambda U(\mathfrak{g})$ is already surjective when restricted to involutions in W_λ; and for type A, it is bijective there. It is therefore natural to ask to what extent this fails in other types.

Conjecture 6.4. The number of elements w of order 2 in W_λ such that $I(w) = I(w_1)$ is the number of irreducible constituents of the W_λ representations $V^L_{w_1}$.

This may even be obvious; at any rate it should be easy.

APPENDIX: Determination of the Double Cells in the Weyl Group of F_4

The Weyl group W of type F_4 has 25 irreducible representations, which we will call $\pi(1),..., \pi(25)$ (or simply $1, 2,...$); we order them as in Alvis's tables [1]. A representation of the Weyl group W_3 of type B_3 (which we regard as a subgroup of W) is parametrized by an order pair (λ_1, λ_2), with λ_1 a partition of p, λ_2 a partition of q, and $p + q = 3$. (This is described for example in [18, p. 326].) Similarly, we regard the Weyl group W_{12} of type $A_1 \times A_2$ as a subgroup of W (with the A_1 factor corresponding to a long root). A representation of W_{12} is also specified by a pair (λ_1, λ_2) with λ_1 a partition of 2 and λ_2 a partition of 3. For the reader's convenience, we reproduce here (Tables AI and AII) Alvis's tables in [26], for restricting a representation of W to W_3 or to W_{12}. Each row corresponds to a representation of W, and each column to the specified representation of the subgroup; the entries are then multiplicities.

We also need to know the tables for restricting to the subgroups of type $A_2 \times A_1$ and C_3. These subgroups are obtained from the previous ones by applying the natural automorphism of F_4 defined by reversing its Coxeter diagram. So it is enough to state how this automorphism, which we will call "flip," permutes the 25 representations of W. We also need to know the result of tensoring with the sign representation, and the lowest degree in which each occurs in $S(\mathfrak{h})$ (Definition 2.14). This information is taken from [1, 5, 19], and tabulated in Table III. (In the last column, we have labelled the special representations. Each non-special one belongs to one of Lusztig's cells (in the sense of [19]) containing the indicated special one.) Finally, we need analogous tables for W_3 and W_{12} (see Tables AIV and AV).

TABLE AI

$W \backslash W_3$	$((3),\phi)$	$((2,1),\phi)$	$((1^3),\phi)$	$(\phi,(3))$	$(\phi,(2,1))$	$(\phi,(1^3))$	$((2),(1))$	$((1^2),(1))$	$((1),(2))$	$((1),(1^2))$
1	1									
2				1						
3			1							
4						1				
5	1			1						
6			1			1				
7		1								
8					1					
9		1			1					
10	1	1					1		1	
11				1	1		1		1	
12		1	1					1		1
13					1	1		1		1
14								1	1	
15							1			1
16							1	1	1	1
17	1						1			
18				1					1	
19			1					1		
20						1				1
21	1			1			1		1	
22			1			1		1		1
23		1					1	1		
24					1				1	1
25		1			1		1	1	1	1

TABLE AII

$W \backslash W_{12}$	$(\square\square, \square\square\square)$	$(\square\square, \begin{smallmatrix}\square\square\\ \square\,\,\,\end{smallmatrix})$	$(\square\square, \begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix})$	$(\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \square\square\square)$	$(\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square\square\\ \square\,\,\,\end{smallmatrix})$	$(\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix})$
1	1					
2			1			
3				1		
4						1
5		1				
6					1	
7	1			1		
8			1			1
9		1			1	
10	2	2		1	1	
11		2	2		1	1
12	1	1		2	2	
13		1	1		2	2
14	1	1			1	1
15		1	1	1	1	
16	1	2	1	1	2	1
17	1	1		1		
18		1	1			1
19	1			1	1	
20			1		1	1
21	1	2	1		1	
22		1		1	2	1
23	2	1		2	1	
24		1	2		1	2
25	1	3	1	1	3	1

We wish to show that each double cell in $\hat{W}$ consists exactly of one special representation, and all the other representations associated to it by Lusztig (Table AIII). The double cell containing the sign representation $\pi(4)$ of W corresponds to the primitive ideal of finite codimension, and so consists of $\pi(4)$ alone. By Proposition 2.25, $\pi(1)$ alone is a double cell. Consider next the double cell of $\pi(17)$. Now $a(\pi(17)) = 1$, and $\pi(17)|_{W_3}$ contains $(\square\square, \square)$. Since $\tilde{a}_{PI}(\square\square, \square) = 1$, Lemma 3.23(c) implies that $\pi(17)$ is a Goldie rank representation. Suppose some other $\pi(k)$ belongs to the same double cell as $\pi(17)$. Then $\tilde{a}_{PI}(\pi(k)) = a(\pi(17)) = 1$; so by Lemma 3.23(b), $\pi(k)|_{W_3}$ cannot contain any σ with $\tilde{a}_{PI}(\sigma) > 1$. By Table A4, this rules out all σ except

$$(\square\square\square, \phi),\quad (\square\square, \square),\quad \left(\begin{smallmatrix}\square\square\\ \square\,\,\,\end{smallmatrix}, \phi\right),\quad \text{and}\quad (\phi, \square\square\square).$$

TABLE AIII

σ	$\dim \sigma$	$a(\sigma)$	$\mathrm{flip}(\sigma)$	$\sigma \otimes \mathrm{sgn}$	Lusztig's special rep.
1	1	0	1	4	SPECIAL
2	1	12	3	3	16
3	1	12	2	2	16
4	1	24	4	1	SPECIAL
5	2	4	7	6	17
6	2	16	8	5	20
7	2	4	5	8	17
8	2	16	6	7	20
9	4	8	9	9	16
10	9	2	10	13	SPECIAL
11	9	6	12	12	16
12	9	6	11	11	16
13	9	10	13	10	SPECIAL
14	6	6	14	14	16
15	6	6	15	15	16
16	12	4	16	16	SPECIAL
17	4	1	17	20	SPECIAL
18	4	7	19	19	16
19	4	7	18	18	16
20	4	13	20	17	SPECIAL
21	8	3	23	22	SPECIAL
22	8	9	24	21	SPECIAL
23	8	3	21	24	SPECIAL
24	8	9	22	23	SPECIAL
25	16	5	25	25	16

The representations $\pi(k)$ whose restriction to W_3 contains only those 4 representations are 1, 2, 5, 7, and 17. By restricting to $W(C_3)$, we can rule out 2 (since $\mathrm{flip}(2) = 3$). We already know that $\pi(1)$ is in a double cell by itself. On the other hand, the set

$$\left\{ \left(\square\!\square, \square \right), \left(\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}, \phi \right) \right\}$$

is a left cell in W_3. By Proposition 3.15, there is a left cell in $\hat{W}$, containing $\pi(17)$, and contained in

$$\mathrm{Ind}_{W_3}^{W} \left(\left(\square\!\square, \square \right) \oplus \left(\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}, \phi \right) \right).$$

TABLE AIV

σ	$\dim\sigma$	$a(\sigma)$	Lusztig's special rep.	$\tilde{a}_{PI}(\sigma)$
$(\ydiagram{3}, \phi)$	1	0	SPECIAL	0
$(\ydiagram{2,1}, \phi)$	2	2	$(\ydiagram{2}, \ydiagram{1})$	1
$(\ydiagram{1,1,1}, \phi)$	1	6	$(\ydiagram{1}, \ydiagram{1,1})$	4
$(\phi, \ydiagram{3})$	1	3	$(\ydiagram{2}, \ydiagram{1})$	1
$(\phi, \ydiagram{2,1})$	2	5	$(\ydiagram{1}, \ydiagram{1,1})$	4
$(\phi, \ydiagram{1,1,1})$	1	9	SPECIAL	9
$(\ydiagram{2}, \ydiagram{1})$	3	1	SPECIAL	1
$(\ydiagram{1,1}, \ydiagram{1})$	3	3	SPECIAL	3
$(\ydiagram{1}, \ydiagram{2})$	3	2	SPECIAL	2
$(\ydiagram{1}, \ydiagram{1,1})$	3	4	SPECIAL	4

TABLE AV

σ	$\dim\sigma$	$a(\sigma)$
$(\ydiagram{2}, \ydiagram{3})$	1	0
$(\ydiagram{2}, \ydiagram{2,1})$	2	1
$(\ydiagram{2}, \ydiagram{1,1,1})$	1	3
$(\ydiagram{1,1}, \ydiagram{3})$	1	1
$(\ydiagram{1,1}, \ydiagram{2,1})$	2	2
$(\ydiagram{1,1}, \ydiagram{1,1,1})$	1	4

This left cell, therefore, cannot contain $\pi(5)$; so it is either $\pi(17)$ or $\pi(17) \oplus \pi(7)$. Now $\pi(17)|_{W_3}$ contains $(\ydiagram{2}, \ydiagram{1})$, but does not contain either

$$\left(\ydiagram{2,1}, \phi\right) \quad \text{or} \quad (\phi, \ydiagram{3}).$$

Therefore it is not a union of left cells for W_3. By Proposition 3.11, $\pi(17)$ cannot be a left cell; so $\pi(17) \oplus \pi(7)$ must be, and $\pi(7)$ belongs to the same double cell as $\pi(17)$. Similarly, $\pi(5)$ belongs to the same double cell as $\pi(17)$. It follows that $\{17, 5, 7\}$ is a double cell. By Proposition 2.25, $\{20, 6, 8\}$ is as well.

Consider next the double cell of $\pi(10)$. We know that it cannot contain the representations 1, 4, 5, 6, 7, 8, 17 or 20 whose double cells we already know. Now $a(\pi(10)) = 2$, $\pi(10)|_{W_3} \supseteq (\ydiagram{1}, \ydiagram{2})$, and $(\ydiagram{1}, \ydiagram{2})$ is special, with $a(\ydiagram{1}, \ydiagram{2}) = 2$. By Lemma 3.23(c), $\pi(10)$ is a Goldie rank representation. Suppose some $\pi(k)$ belongs to the same double cell as $\pi(10)$. By Lemma 3.23(b), $\pi(k)|_{W_3}$ cannot contain any σ with $\tilde{a}_{PI}(\sigma) > 2$. These are

$$\left(\ydiagram{1}, \ydiagram{1,1}\right), \left(\ydiagram{1,1,1}, \phi\right), \left(\phi, \ydiagram{2,1}\right), \left(\phi, \ydiagram{1,1,1}\right), \text{ and } \left(\ydiagram{1,1}, \ydiagram{1}\right).$$

The representations not ruled out in this way, or by being in known double cells, are 2, 10, 15, 18, and 21. Since $\text{flip}(2) = 3$, $\text{flip}(18) = 19$, and $\text{flip}(21) = 23$, it follows that 2, 18, and 21 are ruled out by considering $\pi(k)|_{W(C_3)}$. Finally,

$$\pi(15)|_{W_{12}} \supseteq \left(\ydiagram{2}, \ydiagram{1,1,1}\right);$$

so since

$$a\left(\ydiagram{2}, \ydiagram{1,1,1}\right) = 3,$$

this rules out 15. So $\pi(10)$ is in a double cell by itself. By Proposition 2.25, $\pi(13)$ is as well.

Next, consider the double cell containing $\pi(23)$. We have

$$a(\pi(23)) = 3, \quad \pi(23)|_{W_3} \supseteq \left(\ydiagram{1,1}, \ydiagram{1}\right), \text{ and } \tilde{a}_{PI}\left(\ydiagram{1,1}, \ydiagram{1}\right) = 3.$$

By Lemma 3.23(c), $\pi(23)$ is a Goldie rank representation. If $\pi(k)$ belongs to the same double cells as $\pi(23)$, then (by the argument above) $\pi(23)|_{W_3}$ cannot contain

$$\left(\boxed{}, \begin{array}{|c|}\hline\\\hline\\\hline\end{array}\right),\ \left(\phi, \begin{array}{|c|c|}\hline&\\\hline\\\cline{1-1}\end{array}\right),\ \left(\begin{array}{|c|}\hline\\\hline\\\hline\\\hline\end{array}, \phi\right),\ \text{or}\ \left(\phi, \begin{array}{|c|}\hline\\\hline\\\hline\\\hline\end{array}\right).$$

Since also k cannot be 1, 4, 5, 6, 7, 8, 10, 13, 17, or 20, the only remaining possibilities are 2, 14, 18, 21, and 23. Since flip(2) = 3 and flip(18) = 19, 2 and 18 are ruled out by considering $\pi(k)|_{W(C_3)}$. Finally, $\pi(14)|_{W_{12}}$ contains

$$\left(\begin{array}{|c|}\hline\\\hline\\\hline\end{array}, \begin{array}{|c|}\hline\\\hline\\\hline\\\hline\end{array}\right);$$

so since

$$a\left(\begin{array}{|c|}\hline\\\hline\\\hline\end{array}, \begin{array}{|c|}\hline\\\hline\\\hline\\\hline\end{array}\right) = 4,$$

Lemma 3.23(b) shows that $\pi(14)$ does not belong to the double cell of $\pi(23)$. Since flip(21) = 23, identical considerations show that $\pi(21)$ is a Goldie rank representation, containing at most $\pi(23)$ in its double cell. Since each double cell has exactly one Goldie rank representation, $\{\pi(21)\}$ and $\{\pi(23)\}$ are each double cells. By Proposition 2.25, $\{22\}$ and $\{24\}$ are as well.

Finally, we consider the double cell of $\pi(16)$; we want to show that it contains all of the remaining representations: $\{2, 3, 9, 11, 12, 14, 15, 16, 18, 19, 25\}$. This set is closed under tensoring with sign; so by Proposition 2.25, it is enough to show that each element $\pi(k)$ of it satisfies $\pi(k) \geqslant_{LR} \pi(16)$. Now

$$a(\pi(16)) = 4,\quad \pi(16)|_{W_3} \supseteq \left(\boxed{}, \begin{array}{|c|}\hline\\\hline\\\hline\end{array}\right), \text{ and } \tilde{a}_{PI}\left(\boxed{}, \begin{array}{|c|}\hline\\\hline\\\hline\end{array}\right) = 4.$$

By Lemma 3.23(c), $\pi(16)$ is a Goldie rank representation. Since the double cell of

$$\left(\boxed{}, \begin{array}{|c|}\hline\\\hline\\\hline\end{array}\right)$$

contains

$$\left(\begin{array}{|c|}\hline\\\hline\\\hline\\\hline\end{array}, \phi\right) \quad \text{and} \quad \left(\phi, \begin{array}{|c|c|}\hline&\\\hline\\\cline{1-1}\end{array}\right),$$

Proposition 3.15 shows that if $\pi(k)|_{W_3}$ contains any of these three representations, then $\pi(k) \geqslant_{LR} \pi(16)$. This applies to $k=3$, 12, 15, 16, 19 and 25. Since flip(2) = 3, flip(11) = 12, and flip(18) = 19, the same argument with $W(C_3)$ in place of W_3 applies to $k = 2$, 11, and 18; this leaves 14. Now

$$\pi(16)|_{W_{12}} \supseteq \left(\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}, \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}\right) \qquad \text{and} \qquad \tilde{a}_{PI}\left(\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}, \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}\right) = 4.$$

Therefore, Proposition 3.15 shows that, since

$$\pi(14)|_{W_{12}} \supseteq \left(\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}, \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}\right),$$

we have $\pi(14) \geqslant_{LR} \pi(16)$. This completes the proof of Theorem 2.29 for F_4.

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